B.Sc. Part II (Hons.)
Paper (2)

Exact differential Equations:— A differential equation is said to be exact if it can be derived from its primitive by direct differentiation without taking recourse to transformation such as elimination etc.

Example— The differential equation $xdy + ydx = 0$ is an exact differential equation as it is derived by direct differentiation for its solution the function
$$xy = c$$

Theorem:— The necessary and sufficient condition that the differential equation
$$Mdx + Ndy = 0$$
where M and N are functions of x and y is exact is that
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof:— We shall first of all prove that the condition is necessary.

i.e. we suppose that $Mdx + Ndy = 0$ is exact and we shall prove that
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
Since $Mdx + Ndy = 0$ is given to be exact, therefore $Mdx + Ndy$ is an exact differential of some function say $u = f(x, y)$

i.e. $Mdx + Ndy = du$ ——— (1)

But we know from differential calculus that

$$du = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \text{ --------- (2)}$$

therefore from (1) and (2), we get

$$Mdx + Ndy = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy.$$

It follows therefore

$$\frac{\partial \psi}{\partial x} = M \text{ --------- (3)}$$

And $\frac{\partial \psi}{\partial y} = N$ --------- (4)

Differentiating (3) partially with respect to y and (4) partially w.r.t. x, we get

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial M}{\partial y} \text{ and } \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Assuming that $\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$ we obtain $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus we prove that the condition is ~~sufficient~~. necessary.

Now, we prove that the condition is sufficient.

That is given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we shall prove that

$Mdx + Ndy = d\psi$

Let $\int M dx = \phi \quad \therefore M = \frac{\partial \phi}{\partial x}$

Again $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$$

$N = \frac{\partial \phi}{\partial y} + \psi(y)$ where $\psi(y)$ is a function of y.

Thus we have

$$M\,dx + N\,dy = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \psi(y)\,dy$$
$$= d(\phi) + \psi(y)\,dy$$
$$= d\{\phi + f(y)\} \text{ where } df(y) = \psi(y)$$

Now letting $\phi + f(y) = u$ where u is a function of x and y, we get $M\,dx + N\,dy = du$

Hence the soluti

Example: 1. Solve $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$

Solution: – Here, we have $M = 2x - y + 1$

$N = 2y - x - 1$ so that

$$\frac{\partial M}{\partial y} = -1 \text{ and } \frac{\partial N}{\partial x} = -1 \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given differential equation is exact.

From the equation we have

$$2x\,dx - (y\,dx + x\,dy) + dx + 2y\,dy - dy = 0$$

$$\Rightarrow 2x\,dx - d(xy) + dx + 2y\,dy - dy = 0$$

Hence integrating, we get

$$2 \cdot \frac{x^2}{2} - xy + x + 2 \cdot \frac{y^2}{2} - y = c$$

i.e. $x^2 - xy + x + y^2 - y = c$

which is the required ~~solut~~ solution.

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